SIAM J. FINANCIAL MATH.
Vol. 12, No. 3, pp. 898–911

May Microcredit Lead to Inclusion?*

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Abstract. We consider a Markov-Chain model for a Micro-Finance Institution (MFI) borrower who can be in one of four states: Applicant (A), Beneficiary (B^- or B^+) of a small or a large loan, or Included (I) in the regular banking system. Given the transition matrix we compute the equilibrium and deduce the influence of probability parameters on what is profitable to the borrower within breaking-even constraints of the MFI. We give a general theorem on the expected income flow of a Markov Chain with Income (MCI), that we then apply to our model to determine the constraints emerging from Absence of Strategic Default (ASD) requirements. These not only bound the probabilities of success from above but sometimes also from below.

Key words. micro-credit, Markov Chain with income, financial inclusion, absence of strategic default

AMS subject classifications. 60J10, 60J20, 91-10, 91G40

DOI. 10.1137/20M1342811

1. Introduction.

A microcredit we want to model here consists of small loans to poor entrepreneurs excluded from regular banking. A typical borrower is often a woman so we will use "she" when we wish to describe the questions from "her" point of view. Being poor, she has no access to the regular banking system and may only apply to a Micro-Finance Institution (MFI) that does not require collateral, and not only provides loans but also supervision and coaching, a kind of education for adults. For this, and for supposed bigger default risk, MFI usually charge much higher interest rates than regular banks. Economist Muhammad Yunus [8], [9] and Grameen Bank were given the 2006 Nobel Peace Price for pioneering modern microcredit since the 1970s and helping poor people to get out of poverty in Bangladesh. Microcredit is usually supported as a way of allowing its beneficiaries to leave poverty, even if this hope is often controversial, for instance when there is no clear entrepreneurial project in the loan, as is often the case in agricultural and pastoral activity [6]. In this paper we address a slightly different question, the possibility that microcredit leads to economic inclusion, by offering an opportunity for poor people to eventually have access to the regular banking system to get loans for their small enterprise.

We discuss this question in the framework of what we call Markov Chains with Income (MCI) considering, in addition to the Markov chain dynamic, the Expected Income Flow (EIF) as a measure for a rational behavior of borrowers. This measure takes into account all the possible futures that the borrower may experience. We give a general theorem relating the

*Received by the editors June 3, 2020; accepted for publication (in revised form) May 14, 2021; published electronically July 19, 2021.
https://doi.org/10.1137/20M1342811
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EIF to the transition matrix of the Markov chain, the income function, and an actualization factor. Our MCI model, and especially the computation of its EIF, allows one to check the key hypothesis of Absence of Strategic Default (ASD): a successful borrower has to be better off with the EIF she will have when reimbursing her loan than the one she would have when not doing so. We will see that if increasing the probabilities of success (probability of getting a loan and probabilities of being successful with it) clearly favors inclusion in the financial system, on the contrary the ASD hypothesis leads to upper bounds for these probabilities and also somewhat more unexpected lower bounds that we can interpret and understand in the context of microcredit.

Our approach has been inspired by the paper [7] by Tedeschi where she considers a repeated lending of a fixed amount, provided the borrower reimburses the previous one, and is excluded of any future lending for a fixed number of time periods otherwise. She also introduces the expected income flow, as a function of parameters of the model, such as the probability to get a loan when applying for one, or the probability to be able to reimburse the loan.

O. Khodr et al. [2], [5] explicitly introduced a Markov chain model as a mathematical formalism of Tedeschi's approach. They extend the model from individual loans to the question of group lending and address the issue of strategic default. Similar mathematical models have also been used by Dhib [1] to discuss microcredit activity in the South of Tunisia.

2. Description of the model. In our model we consider an MFI that works for a group of potential borrowers. Its main objective is to help each of them, by granting two successive microloans, to be accepted in the regular banking system if they have been successful, thanks to their positive credit history. When she is granted a microloan, the borrower faces high interest rates thus their small business does not produce a large net income. To model a strong incentive for inclusion, we assume that the small business allows the borrower to cover his basic living costs c but without generating any net income. On the contrary, we assume that when she is Included in the regular banking system, the interest rate is much lower and allows her to generate a net income flow.

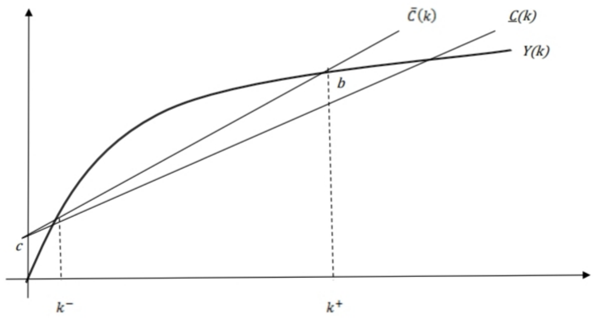

Figure 1. *Production function and cost functions depending on the interest rate of the loan.*

More precisely, let $Y(k)$ be the production function that represents the income the borrower can get from a loan k. Assume $Y(k)$ is increasing and concave. Denote by $\overline{C}(k)$ and $\underline{C}(k)$ the cost functions, $\overline{C}(k) = c + (1 + r)k$, $\underline{C}(k) = c + (1 + r')k$ which are the sum of the basic consumption c of the borrower for one period that we assume to be constant, and the amount she has to reimburse (r and r' represent the interest rates for a microcredit and a regular credit, respectively, $r > r'$). We assume the functions $Y(k)$, $\overline{C}(k)$, and $\underline{C}(k)$ are as in Figure 1, which means that a microcredit (with interest rate r) is only profitable when $k^- \leq k \leq k^+$. We will choose two kinds of microloans, a small one for $k = k^-$ and a bigger one for $k = k^+$. As $Y(k^-) = \overline{C}(k^-)$ and $Y(k^+) = \overline{C}(k^+)$, none of these microloans generate profit beyond covering basic living costs. On the contrary, we assume that when Included in the regular banking system the borrower receives a loan of same size k^+ and, as the interest rate r' is smaller, $r' < r$, the net income $Y(k^+) - \underline{C}(k^+) = b > 0$ is positive, this loan is now profitable, generating a net benefit b, as long as the borrower is successful and thus stays at state I.

To describe the dynamic of the model, we use the following Markov chain. We assume the borrower may be in one of four states: Applicant (A) asking for a small loan, Beneficiary (B^-) of a small loan, Beneficiary (B^+) of a large loan, and Included (I) in the regular banking system. The set of states is $\mathcal{S} := \{A, B^-, B^+, I\}$ and the probabilities to move from one state to another is given in Figure 2.

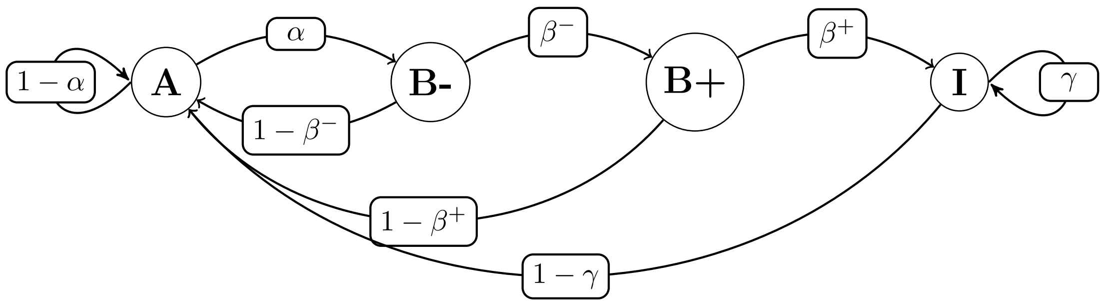

Figure 2. *The four states of the potential borrower (Applicant for a loan (A), Beneficiary of a small loan (B^-), Beneficiary of a large loan (B^+), and Included (I)) and the probabilities to move from one state to another.*

An applicant A may either become a borrower B^- with probability α or stay at state A with probability $1 - \alpha$. We can see α as the proportion of successful applicants in the population targeted by the MFI. This proportion can be small even if the MFI willingly grants loans. A borrower at the state B^- is either successful in her business and able to repay her loan with probability β^- and then able to get a new larger loan k^+, or she can't reimburse her loan and she returns to state A with probability $1 - \beta^-$. Similarly the borrower at the state B^+ is either able to repay her loan and she then becomes Included and gets again a loan k^+ but with a better interest rate r' from the regular banking system or she can't reimburse her loan and she returns to state A with probability $1 - \beta^+$. The two parameters β^- and β^+ can be seen as the probabilities of success of the small business granted by the small k^- and large k^+ microloans. Finally, the borrower in state I is either successful and reimburses her loan, then she will stay in state I and get the same loan again with probability γ, or returns to state A with probability $1 - \gamma$ if she can't reimburse her loan. Denote by P the transition matrix of the Markov chain:

$$P = \begin{pmatrix} 1-\alpha & \alpha & 0 & 0 \\ 1-\beta^- & 0 & \beta^- & 0 \\ 1-\beta^+ & 0 & 0 & \beta^+ \\ 1-\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{2.1}$$

To summarize, the dynamic of the population of potential borrowers is given by a Markov chain with four states and with a transition matrix depending on four parameters, α, β^-, β^+, and γ and to describe the income produced by the business thanks to the different loans, we assume an absence of net income when the borrower gets her loan from an MFI and a positive net income b when she is Included in the regular banking system.

3. The mathematical structure of Markov chain with income.

To better understand the model, it is useful to see it as an example of a general mathematical structure. We call it a Markov Chain with Income (MCI). The idea is to consider not only the stochastic dynamic of an agent moving from one state at time t to another state at time $t+1$, with law given by the Markov chain X, but also an income function f (profit or loss) the agent can get moving along this dynamic and a discount factor δ allowing one to compute the present value of the income flow that the agent can get in the future.

Definition 3.1. *We call Markov Chain with Income (MCI) a triplet (X, f, δ) where*
1. *X is a Markov Chain $(X_t)_{t=0,1,\ldots}$ with finite states space $\mathcal{S} = (S_i)_{1,\ldots,i_{\max}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with transition $i_{max} \times i_{max}$ matrix $P = (p_{ij})_{1\le i,j\le i_{max}}$, $p_{ij} = \mathbb{P}(X_t = S_j \mid X_{t-1} = S_i)$, and initial distribution π_0.*
2. *f is a function $f\ : \mathcal{S}\times\mathcal{S} \longrightarrow \mathbb{R}$, $(S_i, S_j) \mapsto f(S_i, S_j)$ called the* income function.
3. *$\delta \in (0,1)$ is a discount factor.*

Given an MCI (X,f,δ), we can consider the random variable on Ω which represents the present value of the income-flow the agent can get moving along the dynamic of the Markov chain X

Definition 3.2. *Given an MCI (X,f,δ), we call income flow of the MCI the random variable F_X given by*

$$F_X = \delta f(X_0, X_1) + \delta^2 f(X_1, X_2) + \cdots = \sum_{t>0} \delta^t f(X_{t-1}, X_t). \tag{3.1}$$

When the initial distributions π_0 of X is simply $\pi_0(S) = 1$ for some state $S \in \mathcal{S}$, we will denote by w^S the expectation of F_X:

Definition 3.3. *Let (X,f,δ) be an MCI and assume the initial distribution of the Markov chain X is $\pi_0(S) = 1$ for some $S \in \mathcal{S}$. We use Expected Income Flow (EIF) starting at state S and denote by w^S the following conditional expectation:*

$$w^S = \mathbb{E}(F_X \mid X_0 = S),$$

and we denote by $W = {}^t(w^{S_1}, \ldots, w^{S_{imax}})$ the vector of all EIFs starting at states S_i. Similarly, for any $S \in \mathcal{S}$, define $\overline{w}^S$ by

$$\overline{w}^S = \mathbb{E}(f(X_0, X_1) \mid X_0 = S),$$

and by $\overline{W} = {}^t(\overline{w}^{S_1}, \ldots, \overline{w}^{S_{i\max}})$ *the vector of all expected incomes* for the first time step.

We have the following result.

Lemma 3.1 (induction formula for the EIF). *Assume the initial distribution of* X *is such that* $\pi_0(S_i) = 1$ *for some* $S_i \in \mathcal{S}$*. Then*

$$w^{S_i} = \delta \overline{w}^{S_i} + \delta \sum_{1 \le j \le i_{max}} p_{ij} w^{S_j}. \tag{3.2}$$

Proof. We have

$$\begin{aligned} w^{S_i} &= \mathbb{E}\left(\delta f(X_0, X_1) + \sum_{t>1} \delta^t f(X_{t-1}, X_t) \,\middle|\, X_0 = S_i\right) \\ &= \delta \overline{w}^{S_i} + \delta \mathbb{E}\left(\sum_{t>1} \delta^{t-1} f(X_{t-1}, X_t)\right) \\ &= \delta \overline{w}^{S_i} + \delta \sum_{1 \le j \le i_{max}} \mathbb{E}\left(\sum_{t>1} \delta^{t-1} f(X_{t-1}, X_t) \,\middle|\, X_1 = S_j\right) p_{ij} \\ &= \delta \overline{w}^{S_i} + \delta \sum_{1 \le j \le i_{max}} p_{ij} w^{S_j}. \end{aligned}$$

The last equality uses the fact that the Markov chain Y^{S_j} defined on $\Omega_j = \{X_1 = S_j\}$ by $Y_t^{S_j} = X_{t+1}$, thus with initial distribution $\mu_0^{Y^{S_j}}(S_j) = 1$, has the same law as (X, π_0) with $\pi_0(S_j) = 1$. ∎

Theorem 3.1. *Let* (X, f, δ) *be an MCI. The vector of expected income flows* W *is given by*

$$W = \delta(\mathbb{I} - \delta P)^{-1} \overline{W} = \delta \left(\sum_{t=0}^{\infty} (\delta P)^t\right) \overline{W}, \tag{3.3}$$

where P *is the transition matrix of* X *and* $\overline{W} = {}^t(\overline{w}^{S_1}, \ldots, \overline{w}^{S_{i\max}})$ *the vector of expected incomes for the first time step.*

Proof. By the induction formula (3.2) for the EIF we have

$$w^{S_i} = \delta \overline{w}^{S_i} + \delta \sum_{1 \le j \le i_{\max}} p_{ij} w^{S_j}$$

for each state $S_i \in \mathcal{S}$. Thus using vectors W and $\overline{W}$, this can be written $W = \delta \overline{W} + \delta P W$, or $(\mathbb{I} - \delta P) W = \delta \overline{W}$. Now, using the norm $\|P\| = \max_i \{\sum_j |p_{ij}|\}$, as $\|\delta P\| = |\delta| \|P\| = \delta \cdot 1 < 1$, we see that the infinite sum $\sum_{t=0}^{\infty} (\delta P)^t$ converges normally to $(\mathbb{I} - \delta P)^{-1}$, thus Theorem 3.1 is proved. ∎

The following result gives a general formula for the derivatives of the EIFs vector W with respect to its parameters that can be useful when studying sensibility of the model to its parameters.

Corollary 3.1. *Let $M = \mathbb{I} - \delta P$. Assume that the transition matrix P and the vector of expected incomes for the first time step $\overline{W}$ are $\mathcal{C}^1$ functions of some parameter $p \neq \delta$. Then we have*

$$\frac{\partial W}{\partial p} = -\delta M^{-1}\frac{\partial M}{\partial p}M^{-1}\overline{W} + \delta M^{-1}\frac{\partial \overline{W}}{\partial p}, \tag{3.4}$$

$$\frac{\partial W}{\partial \delta} = M^{-1}(\mathbb{I} + \delta P M^{-1})\overline{W}. \tag{3.5}$$

Proof. As $MM^{-1} = \mathbb{I}$, we have $\frac{\partial M}{\partial p}M^{-1} + M\frac{\partial M^{-1}}{\partial p} = 0$, so

$$\frac{\partial M^{-1}}{\partial p} = M^{-1}M\frac{\partial M^{-1}}{\partial p} = -M^{-1}\frac{\partial M}{\partial p}M^{-1}.$$

Formula (3.4) follows then from (3.3) $W = \delta(\mathbb{I} - \delta P)^{-1}\overline{W}$. Moreover, clearly $\frac{\partial M}{\partial \delta} = \frac{\partial}{\partial \delta}(\mathbb{I} - \delta P) = -P$ as P does not depend on δ, so, again from $\mathbb{I} = MM^{-1}$, we have $0 = \frac{\partial M}{\partial \delta}M^{-1} + M\frac{\partial M^{-1}}{\partial \delta} = -PM^{-1} + M\frac{\partial M^{-1}}{\partial \delta}$ thus $\frac{\partial M^{-1}}{\partial \delta} = M^{-1}PM^{-1}$, and thus (3.3) implies

$$\frac{\partial W}{\partial \delta} = \left(M^{-1} + \delta\frac{\partial M^{-1}}{\partial \delta}\right)\overline{W} = (M^{-1} + \delta M^{-1}PM^{-1})\overline{W} = M^{-1}(\mathbb{I} + \delta P M^{-1})\overline{W}.$$

4. Consequences for our model.

4.1. Explicit formulas. We come back to our model for which the state space is $\mathcal{S} = \{A, B^-, B^+, I\}$, and P is given by (2.1). Recall that, for the sake of simplicity, we chose the microloans k such that the value of the production function $Y(k)$ covers the living expenses c of the borrower and the reimbursement $(1 + r)k$ of the loan at the large MFI rate r: the small loan k^- is then chosen to optimize the chances β^- of success (or minimize the risk for the MFI), and the large loan k^+ in order to allow the borrower to show her ability to manage a big loan. Assuming that the amount of this large loan is the same amount as the one given by the MFI, but at a better interest rate $r' < r$, leads to a benefit $b = k^+(r - r')$ for the borrower (see Figure 1) when Included. So the income function is

$$f(S', S'') = \begin{cases} k^+(r - r') & \text{if } S' = I = S'', \\ 0 & \text{in all other cases.} \end{cases} \tag{4.1}$$

Notice that, for any state $S \in \mathcal{S}$, the expected income flow w^S is a function of all parameters $\alpha, \beta^-, \beta^+, \gamma, \delta, k^-, k^+, r, r'$ of the model. Let $M = \mathbb{I} - \delta P$, $\Delta = \det(M)$, and C be the transposed of the co-factors matrix of M so that $M^{-1} = \frac{1}{\Delta}C$. A somewhat lengthy but straightforward computation shows the following proposition.

Proposition 4.1. *For $\delta \in (0, 1)$, the inverse matrix of $M = \mathbb{I} - \delta P$ is given by $M^{-1} = \frac{1}{\Delta}C$, with*

$$C = \begin{pmatrix} 1 - \delta\gamma & \delta\alpha(1 - \delta\gamma) & \delta^2\alpha\beta^-(1 - \delta\gamma) & \delta^3\alpha\beta^-\beta^+ \\ C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} \\ C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} \\ \delta(1 - \gamma) & \delta^2\alpha(1 - \gamma) & \delta^3\alpha\beta^-(1 - \gamma) & C_{4,4} \end{pmatrix} \tag{4.2}$$

with

$$\begin{aligned}
C_{2,1} &= \delta - \delta(1-\delta)\beta^- - \delta^2\gamma - \delta^2(1-\delta)\beta^-\beta^+ + \delta^2(1-\delta)\beta^-\gamma,\\
C_{2,2} &= 1-\delta+\delta\alpha-\delta(1-\delta)\gamma-\delta^2\alpha\gamma,\\
C_{2,3} &= \delta(1-\delta)\beta^- + \delta^2\alpha\beta^- - \delta^2(1-\delta)\beta^-\gamma - \delta^3\alpha\beta^-\gamma,\\
C_{2,4} &= \delta^2(1-\delta)\beta^-\beta^+ + \delta^3\alpha\beta^-\beta^+,\\
C_{3,1} &= \delta - \delta(1-\delta)\beta^+ - \delta^2\gamma,\\
C_{3,2} &= \delta^2\alpha - \delta^2(1-\delta)\alpha\beta^+ - \delta^3\alpha\gamma,\\
C_{3,3} &= 1-\delta+\delta(1-\delta)\alpha - \delta(1-\delta)\gamma + \delta^2\alpha\beta^- - \delta^2(1-\delta)\alpha\gamma - \delta^3\alpha\beta^-\gamma,\\
C_{3,4} &= \delta(1-\delta)\beta^+ + \delta^2(1-\delta)\alpha\beta^+ + \delta^3\alpha\beta^-\beta^+,\\
C_{4,4} &= 1-\delta+\delta(1-\delta)\alpha + \delta^2(1-\delta)\alpha\beta^- + \delta^3\alpha\beta^-\beta^+,
\end{aligned}$$

and

$$\begin{aligned}
\Delta &= \det(\mathbb{I}-\delta P)\\
&= (1-\delta)\left(1+\delta\alpha-\delta\gamma+\delta^2\alpha\beta^- - \delta^2\alpha\gamma + \delta^3\alpha\beta^-\beta^+ - \delta^3\alpha\beta^-\gamma\right) > 0.
\end{aligned} \tag{4.3}$$

So Δ is a degree one polynomial for each parameter α, β^-, β^+, and γ, and so are all coefficients $C_{i,j}$ of matrix C.

Proof. Straightforward computation shows that $\frac{1}{\Delta}C(\mathbb{I}-\delta P) = \mathbb{I} = \frac{1}{\Delta}(\mathbb{I}-\delta P)C$. The fact that $\Delta > 0$ for $\delta \in (0,1)$ follows from the fact that Δ is a continuous function of δ, equal to 1 for $\delta = 0$ and can't vanish on $[0,1)$ as $\Delta = \det(\mathbb{I}-\delta P)$ and $M = \mathbb{I}-\delta P$ is invertible. ■

4.2. Proportions of agent types in the population at equilibrium. The dynamic of most of the Markov chains involved in such models has a useful property: whatever the initial distribution of the agents among the different states, the dynamic modifies the distribution in such a way that when time tends to infinity, this distribution tends to a limit distribution (see Figure 3). This result is a consequence of the Perron–Frobenius theorem that will give proposition (4.2) bellow. We will then draw some consequences in terms of microcredit activity of the MFI.

Proposition 4.2. *Let $(X_t)_{t\geq 0}$ be the above Markov Chain with initial distribution π_0. Assume that all probability parameters α, β^-, β^+, and γ are neither equal to 0 nor 1, so that P is irreducible. Let π_t denote its distribution at time t. Then $\lim_{t\to\infty}\pi_t = \pi^\infty$, where*

$$\pi^\infty = (\pi^\infty_A, \pi^\infty_{B^-}, \pi^\infty_{B^+}, \pi^\infty_I); \pi^\infty(\alpha,\beta^-,\beta^+,\gamma) = \frac{(1-\gamma, \alpha(1-\gamma), \alpha\beta^-(1-\gamma), \alpha\beta^-\beta^+)}{D(\alpha,\beta^-,\beta^+,\gamma)}, \tag{4.4}$$

where $D(\alpha,\beta^-,\beta^+,\gamma) = 1-\gamma+\alpha(1-\gamma)+\alpha\beta^-(1-\gamma)+\alpha\beta^-\beta^+$.

Proof. As P is a stochastic matrix, i.e., positive with line coefficients adding up to 1 and as $P^3 > 0$, the matrix P is primitive and Perron–Frobenius theorem applies. Thus 1 is a left-eigenvalue and P has a unique positive corresponding left-eigenvector π^∞, with coefficients adding up to 1. It is easy to check that the vector π^∞ given by (4.4) is this eigenvector. In addition, we have $\lim_{t\to\infty}\pi_0 P^t = \pi^\infty$ for any initial distribution π^0 and $\lim_{t\to\infty} P^t = P^\infty$ exists, with all lines of P^∞ equal to π^∞. ■

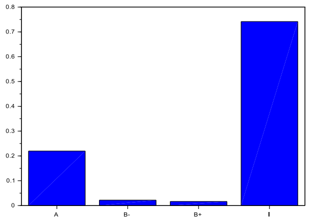

Figure 3. *The equilibrium distribution* $\pi^\infty = (\pi_A^\infty, \pi_{B-}^\infty, \pi_{B+}^\infty, \pi_I^\infty) = \pi^\infty(\alpha, \beta^-, \beta^+, \gamma)$ *between the four states, for* $(\alpha, \beta^-, \beta^+, \gamma) = (0.1, 0.75, 0.9, 0.98)$.

Notice that the limiting proportions $(\pi_A^\infty, \pi_{B-}^\infty, \pi_{B+}^\infty, \pi_I^\infty)$ of applicants and borrowers in the population given by the previous proposition are only reached when t tends to infinity. But when microcredit activity of an MFI is running on a specific group of people for a while, one can consider these proportions as valid (or about to be valid). This is why it can be interesting for an MFI to take them into account when trying to improve its activity. This is the purpose of the following two results.

Proposition 4.3. *The limit proportion* π_I^∞ *of people getting access to regular credit thanks to microcredit and the proportion* $\pi_{B-}^\infty + \pi_{B+}^\infty + \pi_I^\infty$ *of people who can self-finance their activity thanks to a loan (microcredit or regular credit) are both increasing functions of the success probability parameters* α, β^-, β^+, *and* γ.

Proof. From Proposition 4.2 we have

$$\pi_I^\infty = \frac{\alpha\beta^-\beta^+}{(1-\gamma) + (1-\gamma)\alpha + (1-\gamma)\alpha\beta^- + \alpha\beta^-\beta^+},$$

$$\pi_{B-}^\infty + \pi_{B+}^\infty + \pi_I^\infty = 1 - \pi_A^\infty = 1 - \frac{(1-\gamma)}{(1-\gamma) + (1-\gamma)\alpha + (1-\gamma)\alpha\beta^- + \alpha\beta^-\beta^+}.$$

We then observe that both proportions involve homographic functions $\varphi(x) = \frac{ax+b}{cx+d}$ for x equal to α, β^-, β^+, and $(1-\gamma)$, for some positive or zero a, b, c, and d, and with $a = 0$ or $b = 0$. Considering the sign of $\varphi'(x) = (ad - bc)/(cx + d)^2$, it is easy to see that φ is increasing when $b = 0$ and decreasing when $a = 0$. The result then follows easily from that remark considering the eight different functions φ involved. ■

The fact that to favor in the long run inclusion (π_I^∞) or access to self-financed activity $(\pi_{B-}^\infty + \pi_{B+}^\infty + \pi_I^\infty)$, one has to increase the probabilities of success α, β^-, β^+, and γ, is easy to understand. But we will see in the next section that other issues will create limitations on these parameters. Now, let us consider another consequence of Proposition 4.2. We will use the fact that the long run proportion is $\frac{\pi_{B+}^\infty}{\pi_{B-}^\infty}$, which gives an estimation of the long run proportion of small and large loans in the portfolio of the MFI, is equal to β^-. As the risk that the MFI faces on small and large loans is not the same, this allows one to compute a minimal interest rate it has to charge.

The MFI faces different risks with their small loans k^- with success rate β^- and their large loans k^+ with success rate β^+. This leads to a minimal average interest rate r_0 it needs to charge to its borrowers to break even. Just to mention, the actual interest rate r the MFI will charge certainly has to be much larger than r_0 as the interest rate it has to charge takes also into account the MFI's costs for educating and monitoring its customers.

Proposition 4.4. *If the MFI itself borrows at rate ρ, the minimal interest rate r_0 it has to charge to break even is r_0 such that*

$$(1+r_0) = (1+\rho)\frac{1+\beta^-\frac{k^+}{k^-}}{\beta^-(1+\beta^+\frac{k^+}{k^-})} = (1+\rho)\frac{\frac{1}{\beta^-}+\frac{k^+}{k^-}}{1+\beta^+\frac{k^+}{k^-}}. \tag{4.5}$$

Proof. According to (4.4), we have $\frac{\pi^\infty_{B^+}}{\pi^\infty_{B^-}} = \beta^-$. So, for each small loan of amount k^- the MFI gives β^- large loans of amount k^+, so a total capital $k = k^- + \beta^- k^+$, which gives an expected reimbursement $\beta^- k^-(1+r) + \beta^+\beta^- k^+(1+r)$, thus an expected reimbursement rate

$$\beta^0(r) = \frac{\beta^- k^-(1+r) + \beta^-\beta^+ k^+(1+r)}{k^- + \beta^- k^+} = \beta^-(1+r)\frac{k^- + \beta^+ k^+}{k^- + \beta^- k^+}.$$

So, to break even, as the MFI borrows k at interest rate ρ, it then has to charge a minimum interest rate r_0 such that $k(1+\rho) = k\beta^0(r_0)$. Thus $1+\rho = \beta^-(1+r_0)\frac{k^-+\beta^+k^+}{k^-+\beta^-k^+}$, which leads to (4.5). ■

The fact that the minimal rate r_0 is decreasing in the parameters β^- and β^+ is easy to understand. Overly small values of β^- and β^+ (probabilities of success of small and large microloans) may lead to very large values of r_0. For example, in the cases of $\frac{k^+}{k^-} = 4$, $\beta^- = 0.75$, and $\beta^+ = 0.9$, formula (4.5) leads to an $r_0 = 0.20579\dots$.

4.3. Absence of strategic default. Default, in the context of a loan, corresponds to the fact that the borrower does not pay back $k(1+r)$ that was agreed upon for a loan of k granted at rate r. Usually this happens when the borrower faced unexpected difficulties; anyway, the contract between lender and borrower decides also what would happen in this unfortunate case, such as the lender becomes the owner of what was given by the borrower as collateral. The default is called strategic when the borrower could actually pay but prefers to face the consequences that were also agreed upon, as it turns out to be preferable for the rational borrower to default than to reimburse. The context of microfinance involving "poor" (borrower without collateral) and "rich" (lender) makes the question of strategic default quite embarrassing, as often the contract between both parties is not just based on rationality but also on morals. Tedeschi [7] looks for default rules as a substitute for "punishment" for not paying. Considering the expected income flow in this question as counterpart allows us to return to a rational approach. For us *Absence of Strategic Default* (ASD) is a requirement for a model for microfinance. In the context of the model introduced here ASD means that a successful borrower reaching some state S^+ from a state S^- should be better off with w^{S^+} than with keeping the $k(1+r)$ she should have paid back and just have w^A as expected income flow. From a quantitative point of view, ASD can be summarized by the following result.

Proposition 4.5. *ASD holds for a set α, β^-, β^+, γ, δ, k^-, k^+, r, r', $r > r'$ of parameters* *if and only if the following inequalities hold:*

$$w^{B^+} \geq (1+r)k^- + w^A, \tag{4.6}$$

$$w^I \geq (1+r)k^+ + w^A. \tag{4.7}$$

Proof. The two conditions here are just the explicit constraint of ASD for a successful borrower reaching B^+ from B^- and reaching I from B^+. There is no possible strategic default for a successful borrower reaching B^- from A as she did not get yet a loan and thus has nothing to reimburse. Finally, for a successful borrower reaching I from I, ASD requires $w^I \geq (1+r')k^+ + w^A$, which results from (4.7), as $r' < r$. ∎

In Proposition 4.3 we have seen that, to favor inclusion (i.e., to increase π_I^∞) or to favor access to self-financed activity (i.e., to increase $\pi_{B^-}^\infty + \pi_{B^+}^\infty + \pi_I^\infty$) it suffices to increase, as much as possible, any of the probability of success α, β^-, β^+, or γ, the upper limitation having to emerge from elsewhere. The next proposition deals with the question of whether ASD creates upper, or lower, limitations in some or all of these parameters. To that purpose, let us define the two following functions ASD^- and ASD^+:

$$\text{ASD}^-(\alpha, \beta^-, \beta^+, \gamma) = w^{B^+} - w^A - (1+r)k^-, \tag{4.8}$$

$$\text{ASD}^+(\alpha, \beta^-, \beta^+, \gamma) = w^I - w^A - (1+r)k^+, \tag{4.9}$$

the first, ASD^-, having to stay nonnegative to avoid strategic default on a small microloan k^-, the second, ASD^+, having to stay nonnegative to avoid strategic default on a large microloan k^+. The following proposition observes that these two functions are monotonic with respect to each of the variables α, β^-, β^+, or γ. The question of whether there is an upper or lower limitation depends on whether these functions are decreasing or increasing. Of course, this also requires that other parameters $(\delta, k^-, k^+, r, r')$ are such that their exists at least one $p_* = (\alpha_*, \beta_*^-, \beta_*^+, \gamma_*)$ such that both $\text{ASD}^-(p_*)$ and $\text{ASD}^+(p_*)$ are positive.

In Figures 4 and 5 we used $\delta = 0.95$, $k^- = 1$ (the choice of numéraire), $k^+ = 5$, $r = 0.20$, $r' = 0.04$, and found a $p_* = (\alpha_*, \beta_*^-, \beta_*^+, \gamma_*) = (0.1, 0.75, 0.9, 0.98)$.

Proposition 4.6. *The functions ASD^- and ASD^+ are well defined on $[0,1]^4$ and all their partial functions are monotonic on $[0,1]$. Both functions are decreasing for α and β^-, so Absence of Strategic Default (ASD) may lead to an upper limitation on these variables. They are both increasing functions of γ, so ASD may lead to a lower limitation on γ. With β^+, function ASD^- is increasing and ASD^+ is decreasing, so ASD may require β^+ to stay in an interval.*

Proof. By Theorem 3.1 and Proposition 4.1, using the definition of the income function f given at (4.1), we have

$$W = \frac{\delta}{\Delta} C \ {}^t(0,0,0,\mathbb{E}(f(X_0,X_1) \mid X_0 = I)) = \frac{\delta(r-r')k^+\gamma}{\Delta} C\, e_4, \text{ where } e_4 = {}^t(0,0,0,1),$$

so we have the following explicit values of functions ASD^- and ASD^+:

$$ASD^-(\alpha, \beta^-, \beta^+, \gamma) = \delta(r-r')k^+ \frac{\gamma(C_{3,4} - C_{1,4})}{\Delta} - (1+r)k^- \text{ and}$$

$$ASD^+(\alpha, \beta^-, \beta^+, \gamma) = \delta(r-r')k^+ \frac{\gamma(C_{4,4} - C_{1,4})}{\Delta} - (1+r)k^+.$$

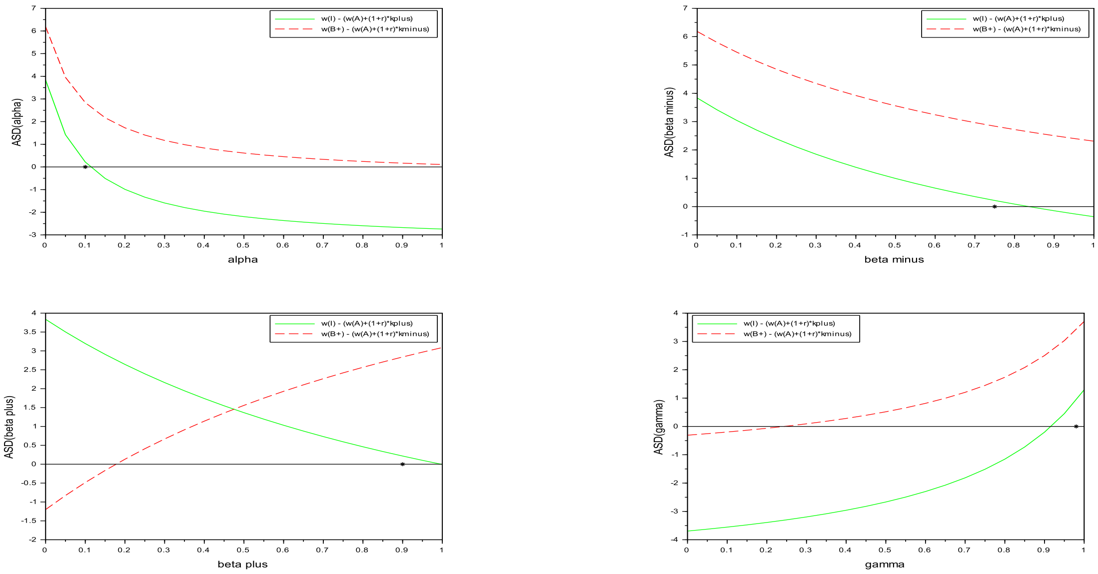

Figure 4. *Check of ASD. Here we chose* $p_* = (\alpha_*, \beta_*^-, \beta_*^+, \gamma_*) = (0.1, 0.75, 0.9, 0.98)$ *for which ASD is granted, and we have plotted the partial functions with value* $w^{B^+} - (1+r)k^- - w^A$ *(dashed line, ASD for a small microloan* k^-*) and* $w^I - (1+r)k^+ - w^A$ *(solid line, ASD for a large microloan* k^+*) which have to stay positive, when changing one of the coordinates. We see that to guarantee ASD one can lower* α *and* β^-*, increase* γ*, but* β^+ *has to stay between a lower and an upper limit.*

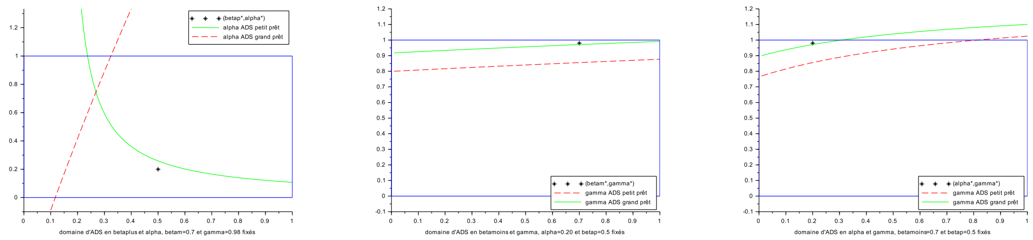

Figure 5. *Border of the intersection of the ASD-domain of* $[0,1]^4$ *with the three planes with equations* $(\beta^-, \gamma) = (\beta_*^-, \gamma_*)$, $(\beta^+, \alpha) = (\beta_*^+, \alpha_*)$, *and* $(\alpha, \gamma) = (\alpha_*, \gamma_*)$*, the star-marked point being in the domain. Dashed line expresses ASD for a small microloan* k^- *and solid line expresses ASD for a large microloan* k^+.

As the matrix P is stochastic and $0 \le \delta < 1$, $\Delta = \det(\mathbb{I} - \delta P) > 0$ for any $(\alpha, \beta^-, \beta^+, \gamma) \in [0,1]^4$, and by explicit formulas (4.2) and (4.3) we see that both ASD$^-$ and ASD$^+$ are homographic functions of each variable α, β^-, β^+, and γ, with pole only at $\Delta = 0$, so outside $[0,1]^4$. Thus all of these partial functions are monotonic decreasing or increasing, depending on whether the difference of their values at 1 and at 0 is negative or positive. Thus we just have to check the sign of ASD$^\pm(\alpha = 1) - $ ASD$^\pm(\alpha = 0)$, and similarly for β^-, β^+, and γ.

Here for α and ASD^-. Let $N^- = \gamma(C_{3,4} - C_{1,4}) = \delta(1-\delta)\beta^+\gamma(1+\delta\alpha)$. We have

$$\begin{aligned}\mathrm{ASD}^-(\alpha=1) - \mathrm{ASD}^-(\alpha=0) &= \delta(r-r')k^+\left[\frac{N^-(\alpha=1)}{\Delta(\alpha=1)} - \frac{N^-(\alpha=0)}{\Delta(\alpha=0)}\right]\\ = \delta(r-r')k^+[N^-(\alpha=1)\Delta(\alpha=0) &- N^-(\alpha=0)\Delta(\alpha=1)]/\Delta(\alpha=0)\Delta(\alpha=1).\end{aligned}$$

As already mentioned, $\Delta > 0$, so the denominator $\Delta(\alpha=0)\Delta(\alpha=1)$ is positive and so is $\delta(r-r')k^+$, so the sign of $\mathrm{ASD}^-(\alpha=1) - \mathrm{ASD}^-(\alpha=0)$ is just the sign of $N^-(\alpha=1)\Delta(\alpha=0) - N^-(\alpha=0)\Delta(\alpha=1)$. Elementary computation shows that this difference is $-\delta^3(1-\delta)^2\beta^-\beta^+\gamma(1+\beta^+-\delta\gamma)$, so it is negative and thus ASD^- is a decreasing function of α.

Similarly, with $N^+ = \gamma(C_{4,4}-C_{1,4}) = \gamma(1-\delta+(1-\delta)\alpha+\delta^2(1-\delta)\alpha\beta^-)$ and now the sign of $\mathrm{ASD}^+(\alpha=1) - \mathrm{ASD}^+(\alpha=0)$ is just the sign of $[N^+(\alpha=1)\Delta(\alpha=0) - N^+(\alpha=0)\Delta(\alpha=1)]$. Elementary computation shows that this difference is $-\delta^3(1-\delta)\beta^-\beta^+\gamma$, so it is negative and thus ASD^+ is a decreasing function of α, too.

By reasoning in the same way for the three others parameters β^-, β^+, and γ, we understand that the other statements in the proposition follow from similar elementary computations, namely

$$\begin{aligned}
&N^-(\beta^-=1)\Delta(\beta^-=0) - N^-(\beta^-=0)\Delta(\beta^-=1)\\
&\qquad = -\delta^3(1-\delta)\alpha(1+\delta\alpha)\beta^+\gamma(1+\delta(\beta^+-\gamma)) < 0,\\
&N^+(\beta^-=1)\Delta(\beta^-=0) - N^+(\beta^-=0)\Delta(\beta^-=1)\\
&\qquad = -\delta^2(1-\delta)\alpha\gamma[\alpha(1-\delta)(1-\delta\gamma) + \delta\beta^+(1-\alpha)] < 0,\\
&N^-(\gamma=1)\Delta(\gamma=0) - N^-(\gamma=0)\Delta(\gamma=1)\\
&\qquad = +\delta(1-\delta)(1+\delta\alpha)\beta^+\Delta(\gamma=0) > 0,\\
&N^+(\gamma=1)\Delta(\gamma=0) - N^+(\gamma=0)\Delta(\gamma=1)\\
&\qquad = +(1-\delta)(1+\alpha+\delta^2\alpha\beta^-)\Delta(\gamma=0) > 0,\\
&N^-(\beta^+=1)\Delta(\beta^+=0) - N^-(\beta^+=0)\Delta(\beta^+=1)\\
&\qquad = \delta(1-\delta)(1+\delta\alpha)\Delta(\beta^+=0) > 0,\\
&N^+(\beta^+=1)\Delta(\beta^+=0) - N^+(\beta^+=0)\Delta(\beta^+=1)\\
&\qquad = -\delta^3(1-\delta)\alpha\beta^-\gamma(1+\alpha+\delta^2\alpha\beta^-) < 0.
\end{aligned}$$

The fact that ASD is a constraint antagonistic to the wish of the MFI to favor inclusion or self-microfinancing is in line with regular-banking "common sense": an overly large chance of getting a microloan (parameter α) or an overly large chance of succeeding in a microloan (parameters β) brings an overly large incentive to default when there is no counterpart. The previous result just provides some quantitative upper limit for these parameters as a function of the various parameters of the model.

More interesting is to understand how a lower-limit can occur. Let us consider first the case when ASD^- creates a lower limit for β^+: so we have to understand why a rational beneficiary of a small loan may prefer to keep for her the $(1+r)k^-$ (and w^A) than becoming B^+ and getting its w^{B^+}, i.e., getting a large microloan k^+. The fact that there is here a lower limit for β^+ just shows that this rational borrower needs to have enough chances to succeed in this new large-loan enterprise. If not, she is better-off with keeping the $(1+r)k^-$ she would be able to pay and apply again for a future possible small loan: a rational microentrepreneur needs to have enough chances to succeed in her business to get a chance, after one more successful step, to be Included in the regular banking system thanks to microcredit. The same happens (with even a higher lower-limit) for a rational (micro)borrower-with-no-counterpart of k^+: If she has too little chance γ to succeed when getting a large regular-banking loan, even when having succeeded ("by chance," possibly), then if she is just rational, she should keep the $(1 + r)k^+$ for her (and w^A) when she is entitled to become Included rather than to pay back her loan. This phenomenon may be a reason why microcredit has not had more success in leading to the regular banking system. This should encourage MFI to concentrate even more on improving the chances of success in their enterprise for beneficiaries of a large loan, by providing education and monitoring, in order to improve personal skills, which increase beneficiaries' chances of success, here (β^+), and also later in their enterprise when Included (γ).

5. Discussion and conclusion.

In this paper we have modeled the successive loans that a microentrepreneur, who initially has no access to a regular banking system, can get thanks to the action of an MFI. For this we have considered an elementary four-state Markov chain which reduces the real situations to four possible states: potential applicant (A), beneficiary of a microloan, first small (B^-) and then larger (B^+), and finally a regular-banking loan when Included (I) in this banking system, with a risk at any time to drop back in the applicant state when not reimbursing a loan, because of an economic accident or just strategic default, if preferable for her. The simplicity of the model allows one to compute explicitly the equilibrium distribution and then check that the better the chances to reach the next step, the better it is for increasing the fraction of microentrepreneurs who finally get access to regular banking. Usually one considers that the main drawback for a potential microentrepreneur to get a loan is that they can't offer any collateral. Generalizing an idea of Tedeschi [7] we consider the drop of expected income flow EIF as the measure of what that the borrower would lose in case of default. We propose a general concept of Markov Chain with Income (MCI) and give a general theorem for computing this EIF as a function of the present state, a theorem that can be applied to much more general situations or finer models. It allows one to compute the EIF explicitly and understand that strategic default emerges not only from overly easy access to microloans, but also from having too little a chance at success in the microenterprise. In our views, this should encourage rational MFI to provide education and monitoring to their borrowers. Clearly, the model we offer can be extended in the MCI framework to become more realistic. The general results given here allow one to check the influence of any change in the model, especially on the effect favoring access to regular banking thanks to microcredit. Of course this modelling research would largely benefit from a Randomized Controlled Trials approach [3], [4] to make the model and its various parameters not only realistic but also useful.

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